

GENERAL MOMENT THEOREMS FOR NON-DISTINCT UNRESTRICTED PARTITIONS

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ABSTRACT. A well-known result from Hardy and Ramanujan gives an asymptotic expression for the number of possible ways to express an integer as the sum of smaller integers. In this vein, we consider the general partitioning problem of writing an integer n as a sum of summands from a given sequence Λ of non-decreasing integers. Under suitable assumptions on the sequence Λ , we obtain results using associated zeta-functions and saddle-point techniques. We also calculate higher moments of the sequence Λ as well as the expected number of summands. Applications are made to various sequences, including those of Barnes and Epstein types. These results are of potential interest in statistical mechanics in the context of Bose-Einstein condensation.

1. INTRODUCTION

The first significant ideas dealing with the theory of partitions can be attributed to Euler [16] and were published in 1748. The next important milestone of partition theory was laid in 1918, when Hardy and Ramanujan [17], using quite involved combinatorics, produced their celebrated theorem, which gives the asymptotic result for the number of ways to express an integer as the sum of lesser integers.

In the mid-1970's, the area of asymptotic analysis blossomed with two of its primary works, those of Dingle [6] and Olver [28]. Among the plethora of information in these two books are some nice results concerning the method of steepest descent, more commonly called the saddle-point method. Also at this time, what is now one of the primary texts of partition theory, Andrews', *The Theory of Partitions* [3], became available. In 1975-76, Richmond used ideas of generating functions [24] and saddle-point methods to describe the moments of certain types of partitions [29, 30]. In the first of this two part series, *The Moments of Partitions I*, using asymptotic analysis instead of combinatorics, Richmond reproduced the results of Hardy and Ramanujan [17] as well as calculating higher moments and variance. In his second paper, *The Moments of Partitions II*, Richmond went on to give asymptotic results for the moments of a general sequence whose associated zeta-function has only one singularity in the interval $(0, 1]$.

Though Richmond's results are remarkable, he does not consider the situation where the sequence gives rise to a zeta-function with arbitrarily many singularities at arbitrary values. Building upon Richmond's results, and taking full advantage of the saddle-point method as well as asymptotic analysis, the present paper addresses that question.

As literature on zeta-functions is readily available, we give only those definitions and propositions which are used. Background on the Riemann zeta-function can be

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found in [9] and [32]. Though the Barnes [4, 5] and Epstein zeta-functions [14, 15] are less well-known, they are of paramount importance in mathematical physics; see, e.g., Actor [1, 2], Dowker [7, 8], Elizalde [10, 11, 13] and Kirsten [22, 23], for a full reckoning. Details of a general zeta-function are given, e.g., by Voros [33].

After presenting the background, we derive what we call the General Moment Theorems. The first of these theorems addresses the question: given a sequence Λ of non-decreasing integers, with limited assumptions, how many ways are there to write a large natural number n as a sum of members from the sequence Λ ? The other General Moment Theorem addresses higher moments. We then apply the General Moment Theorems to find the expected number of summands.

We make applications to a variety of sequences, starting with recreating the results of Richmond's first paper, among other things re-deriving the famous Hardy and Ramanujan theorem described above. The General Moment Theorems are then applied to a sequence whose associated zeta-function has only one singularity. In essence, this is what Richmond describes in his second paper, except we assume no restriction on the location of the singularity. We then reproduce the results of Nanda [25] for two and three-dimensional Barnes-type partitions, and proceed to consider two and three-dimensional Epstein-type partitions.

The advent of mathematical physics, specifically the area of statistical mechanics in the context of Bose-Einstein condensation, has brought partition theory to the forefront of current research, see, e.g., [19, 20, 26]. It is because of this context that we require the sequence Λ to consist of non-decreasing integers instead of increasing integers. More specifically, we allow the sequence to contain the same integer a finite number of times. In the physical context mentioned the repetition of the same number represents *different* quantum mechanical states with the *same* energy, so-called degenerate states. Distributing a given amount of energy among the states corresponds exactly to the partitioning problem considered here, where the repeated numbers correspond to the different states with the same energy. We hope that the results of this article will be of interest to not only the physics community, but to number theorists as well.

2. ZETA-FUNCTIONS

2.1. Basic zeta-functions. In this section, we give the definitions and some properties of specific zeta-functions needed for use in later sections.

Definition 1. Let $s \in \mathbb{C}$ with $\Re s > 1$. The Riemann zeta-function is defined as

$$\zeta_{\mathcal{R}}(s) = \sum_{n \in \mathbb{N}} \frac{1}{n^s}.$$

It is well known that a meromorphic extension of $\zeta_{\mathcal{R}}(s)$ to the whole complex plane can be constructed and that the only pole of $\zeta_{\mathcal{R}}(s)$ is at $s = 1$ [9].

Definition 2. Let $s \in \mathbb{C}$ with $\Re s > d$, and $c \in \mathbb{R}$, $\vec{r} \in \mathbb{R}^d$ such that $c + \vec{m}\vec{r} > 0$ for all $\vec{m} \in \mathbb{N}_0^d$. The Barnes zeta-function is defined as [4, 5]

$$\zeta_{\mathcal{B}}(s, c | \vec{r}) = \sum_{\vec{m} \in \mathbb{N}_0^d} \frac{1}{(c + \vec{m}\vec{r})^s}.$$

For $c = 0$ it will be understood that the summation only runs over $\vec{m} \in \mathbb{N}_0^d - \{\vec{0}\}$.

For $\vec{r} = (1, 1, \dots, 1, 1) := \vec{1}$ we will use the notation $\zeta_{\mathcal{B}}(s, c) := \zeta_{\mathcal{B}}(s, c|\vec{1})$, in which case we have the following expansion [4, 5]:

Proposition 1. *Let $\vec{r} = \vec{1}$ and $c > 0$, then*

$$\zeta_{\mathcal{B}}(s, c) := \zeta_{\mathcal{B}}(s, c|\vec{1}) = \sum_{l=0}^{\infty} e_l^{(d)} (c+l)^{-s},$$

where

$$e_l^{(d)} = \binom{l+d-1}{d-1}.$$

In the case $\vec{r} = \vec{1}$, the above result indicates that the Barnes zeta-function can be represented in terms of the so-called Hurwitz zeta-function.

Definition 3. Let $s \in \mathbb{C}$ with $\Re s > 1$ and $c \in \mathbb{R}$ with $c > 0$. The Hurwitz zeta-function is defined as

$$\zeta_H(s, c) = \sum_{n=0}^{\infty} \frac{1}{(n+c)^s}.$$

The Hurwitz zeta-function can be meromorphically continued to the whole complex plane and its only pole is located at $s = 1$.

Barnes showed the following relation between the Hurwitz and Barnes zeta-function [4, 5]:

Corollary 1. *The Barnes zeta-function $\zeta_{\mathcal{B}}(s, c)$ can be meromorphically continued to the whole complex plane and for $c \in \mathbb{R}$ with $c > 0$ one has*

$$\zeta_{\mathcal{B}}(s, c) = \sum_{k=1}^d \frac{(-1)^{k+d}}{(k-1)!(d-k)!} B_{d-k}^{(d)}(c) \zeta_H(s+1-k, c).$$

Here $B_i^{(d)}(c) := B_i^{(d)}(c|\vec{1})$ are the generalized Bernoulli polynomials [27] defined by

$$\frac{e^{-ct}}{\prod_{j=1}^d (1 - e^{-r_j t})} = \frac{(-1)^d}{\prod_{j=1}^d r_j} \sum_{n=0}^{\infty} \frac{(-t)^{n-d}}{n!} B_n^{(d)}(c|\vec{r}).$$

Similar results can be derived for the case $c = 0$. For example in dimension $d = 2$ one finds

$$(1) \quad \zeta_{\mathcal{B}_2}(s, 0) = \zeta_{\mathcal{R}}(s-1) + \zeta_{\mathcal{R}}(s),$$

whereas in dimension $d = 3$ the answer reads

$$\zeta_{\mathcal{B}_3}(s, 0) = \frac{1}{2} (\zeta_{\mathcal{R}}(s-2) + 3\zeta_{\mathcal{R}}(s-1) + 2\zeta_{\mathcal{R}}(s)).$$

Finally we will consider sequences related to sums of squares of integers; the following zeta-function will be useful.

Definition 4. Define $Q(\vec{m}, \vec{r}) = r_1 m_1^2 + r_2 m_2^2 + \dots + r_d m_d^2$. Let $s \in \mathbb{C}$ with $\Re s > \frac{d}{2}$, and $c \in \mathbb{R}$, $\vec{r} \in \mathbb{R}^d$ such that $c + Q(\vec{m}, \vec{r}) > 0$ for all $\vec{m} \in \mathbb{N}_0^d$. The Epstein zeta-function [14, 15] is defined as

$$\zeta_{\mathcal{E}}(s, c|\vec{r}) = \sum_{\vec{m} \in \mathbb{N}_0^d} \frac{1}{(c + Q(\vec{m}, \vec{r}))^s}.$$

In case $c = 0$, it is understood that the summation ranges over $\vec{m} \in \mathbb{N}_0^d - \{\vec{0}\}$ only.

For all cases meromorphic continuations of $\zeta_{\mathcal{E}}(s, c|\vec{r})$ can be constructed; see, e.g., [13, 14, 15, 22]. Because of the complicated appearance, involving series over Bessel functions, we do not display them explicitly.

2.2. A general zeta-function. In this section we first introduce the general type of sequences we want to use, giving them certain basic restrictions, then we examine residues and particular values of the zeta-function associated with the sequence.

Throughout this paper we denote by Λ a sequence that is a nondecreasing sequence of natural numbers such that the following hold.

- (i) $1 \in \Lambda$.
- (ii) The partition function,

$$(2) \quad \Theta(t) = \sum_{\lambda \in \Lambda} e^{-\lambda t},$$

converges for $t > 0$.

- (iii) For $t \rightarrow 0^+$, $\Theta(t)$ admits a full asymptotic expansion

$$(3) \quad \Theta(t) \sim \sum_{n \in \mathbb{N}_0} A_{i_n} t^{i_n},$$

where $i_n \in \mathbb{R}$ with $i_{n+1} > i_n$, $i_0 < 0$ and where $i_n \rightarrow \infty$ as $n \rightarrow \infty$. Later on, we will occasionally also use the notation $-i_n = \mu_n$.

The fact that we restrict Λ to be a sequence of natural numbers is due to the fact that we analyze partitioning problems of natural numbers. In writing down the restrictions (ii) and (iii) we follow [33], where it is shown that these requirements lead to the well-defined spectral functions considered in the following. In particular, with these restrictions on the sequence Λ of numbers, we are now in a position to define a general zeta-function (of Λ -type) as the following.

Definition 5. Let Λ be a sequence as described above, and let $s \in \mathbb{C}$ with $\Re s > \mu_0$. We define the general Λ -type zeta-function as

$$\zeta_{\Lambda}(s) = \sum_{\lambda \in \Lambda} \frac{1}{\lambda^s}.$$

Typically, zeta functions are built from eigenvalues of an elliptic (pseudo) differential operator [11, 31], but it remains a perfectly viable spectral function in the given context [33].

For our later considerations, we will only need to know residues and particular values of ζ_{Λ} . To find these, we will use the standard integral representation

$$\zeta_{\Lambda}(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \Theta(t) dt.$$

For the residues, and values of $\zeta_{\Lambda}(s)$ at $s = -n$, only the small- t behavior of the integrand is relevant and we may focus our examination on the function

$$L(s) = \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \Theta(t) dt,$$

realizing that the residues, and values of $\zeta_{\Lambda}(s)$ at $s = -n$, are precisely those of $L(s)$.

Formally substituting (3) into the above equation and performing the integration yields the expression

$$L(s) = \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \sum_{n \in \mathbb{N}_0} A_{i_n} t^{i_n} dt = \frac{1}{\Gamma(s)} \sum_{n \in \mathbb{N}_0} A_{i_n} \frac{1}{s + i_n}.$$

This formal calculation can be made precise and so allows to conclude the following propositions [31, 33]:

Proposition 2. *For $i_n \neq k \in \mathbb{N}_0$, the residues of $\zeta_\Lambda(s)$ occur at $s = -i_n$, and furthermore,*

$$(4) \quad \operatorname{Res}_{s=-i_n} \{\zeta_\Lambda(s)\} = \frac{A_{i_n}}{\Gamma(-i_n)}.$$

Proposition 3. *For $n \in \mathbb{N}_0$,*

$$\zeta_\Lambda(-n) = (-1)^n n! A_n.$$

3. GENERAL MOMENT THEOREMS

3.1. General results for all k . We start this section by defining moments of partitions.

Definition 6. Let $p_\Lambda(n, m)$ be the number of partitions of n into m summands where each summand is a member of Λ . The k -th moment of $p_\Lambda(n, m)$ is denoted by $t_\Lambda^k(n)$, and is defined by

$$t_\Lambda^k(n) = \sum_{m \in \mathbb{N}_0} m^k p_\Lambda(n, m),$$

where $p_\Lambda(n, m) = 0$ for $m > n$ and $p_\Lambda(n, 0) = 0$.

For small n , the above definition is sufficient for finding values of $t_\Lambda^k(n)$ by hand or using a little computer program. For large n , the calculation is much more difficult; we must find a more reasonable way to compute $t_\Lambda^k(n)$. We resolve this problem by first constructing a generating function, then we continue by evaluating the coefficients of this generating function; see [24, 29].

Let us start by defining

$$(5) \quad G_\Lambda(x, z) = \prod_{\lambda \in \Lambda} (1 - zx^\lambda)^{-1} = \sum_{n \in \mathbb{N}_0} \sum_{m \in \mathbb{N}_0} p_\Lambda(n, m) x^n z^m,$$

and

$$\vartheta = z \frac{\partial}{\partial z}.$$

Note that

$$\vartheta^k G_\Lambda(x, z) \Big|_{z=1} := \vartheta^k G_\Lambda(x) = \sum_{n \in \mathbb{N}_0} \sum_{m \in \mathbb{N}_0} m^k p_\Lambda(n, m) x^n = \sum_{n \in \mathbb{N}_0} t_\Lambda^k(n) x^n,$$

so that we have constructed a generating function for $t_\Lambda^k(n)$.

Since Λ contains only integers, only $n \in \mathbb{N}_0$ occurs in the summation. We may therefore apply Cauchy's formula for Laurent series coefficients, so that for $\varepsilon > 0$ suitably chosen, we have

$$t_\Lambda^k(n) = \frac{1}{2\pi i} \int_{C(0, \varepsilon)} \vartheta^k G_\Lambda(x) x^{-(n+1)} dx,$$

where $C(0, \varepsilon)$ is the circle of radius ε about $x = 0$. With the substitution $x = e^{-a}$, this easily becomes

$$(6) \quad t_\Lambda^k(n) = \frac{1}{2\pi i} \int_{s.p.} e^{n(a + \frac{1}{n} \log \vartheta^k G_\Lambda(e^{-a}))} da,$$

where $s.p.$ indicates a closed path that goes through the saddle-point $a = \alpha_k$ of the integrand.

The saddle-point $a = \alpha_k$ is found as a solution to the equation

$$(7) \quad \frac{d}{da} \left(a + \frac{1}{n} \log \vartheta^k G_\Lambda(e^{-a}) \right) = 0.$$

As we will see in the following, the large- n expansion of the moments results from a small- $|a|$ expansion of the saddle-point equation.

In order to evaluate (6) as $n \rightarrow \infty$, that is, as $|a| \rightarrow 0$, it will be necessary to find a more explicit form of the saddle-point equation. We first simplify $\vartheta^k G_\Lambda(e^{-a})$, making two cases, namely $k = 1$ and $k \geq 2$. For $k = 1$,

$$\begin{aligned} \vartheta G_\Lambda(x, z) &= z \frac{\partial}{\partial z} \left\{ \prod_{\lambda \in \Lambda} (1 - zx^\lambda)^{-1} \right\} \\ &= z \left(\sum_{\lambda \in \Lambda} \frac{x^\lambda}{1 - zx^\lambda} \right) \left(\prod_{\lambda \in \Lambda} (1 - zx^\lambda)^{-1} \right) \\ &= G_\Lambda(x, z) \sum_{\lambda \in \Lambda} \frac{zx^\lambda}{1 - zx^\lambda}. \end{aligned}$$

For convenience we denote

$$(8) \quad S_\Lambda(x, z) = \sum_{\lambda \in \Lambda} \frac{zx^\lambda}{1 - zx^\lambda} \quad \text{and} \quad \vartheta^k S_\Lambda(x, z)|_{z=1} := \vartheta^k S_\Lambda(x).$$

From the above string of equalities, we have $\vartheta G_\Lambda(e^{-a}) = G_\Lambda(e^{-a}) S_\Lambda(e^{-a})$.

Repeating the above process for $k \geq 2$ yields [29]

$$\vartheta^k G_\Lambda(e^{-a}) = G_\Lambda(e^{-a}) S_\Lambda^{(k)}(e^{-a}),$$

with

$$(9) \quad S_\Lambda^{(k)}(e^{-a}) = \sum \frac{k!}{b_1! b_2! \cdots b_k!} \left(\frac{\vartheta^0 S_\Lambda(e^{-a})}{1!} \right)^{b_1} \times \left(\frac{\vartheta^1 S_\Lambda(e^{-a})}{2!} \right)^{b_2} \times \cdots \times \left(\frac{\vartheta^{k-1} S_\Lambda(e^{-a})}{k!} \right)^{b_k},$$

where the summation is over all solutions $b_1, b_2, \dots, b_k \in \mathbb{N}_0$ of $b_1 + 2b_2 + \cdots + kb_k = k$ (see [21] for more details on sums like (9)). For readability, we denote the summation in (9) by $\sum(\vartheta^0 S_\Lambda(e^{-a}), \vartheta^1 S_\Lambda(e^{-a}), \dots, \vartheta^{k-1} S_\Lambda(e^{-a}))$. Using the above quantities, the saddle-point equation (7) reads

$$\frac{d}{da} \left(a + \frac{1}{n} \log G_\Lambda(e^{-a}) + \frac{1}{n} \log S_\Lambda^{(k)}(e^{-a}) \right) = 0,$$

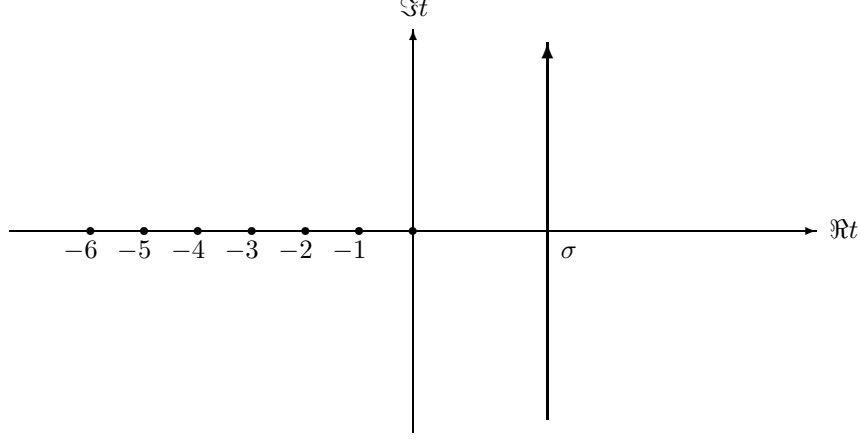


FIGURE 1. The contour from $\sigma - i\infty$ to $\sigma + i\infty$ in the complex plane.

or, more explicitly,

$$(10) \quad n = \sum_{\lambda \in \Lambda} \frac{\lambda}{e^{a\lambda} - 1} - \frac{\frac{d}{da} S_{\Lambda}^{(k)}(e^{-a})}{S_{\Lambda}^{(k)}(e^{-a})}.$$

We next show that for large n the solution to this saddle-point equation is unique.

At this point, we begin to make systematic use of the following easily shown identity.

Proposition 4. *Let $\sigma > 0$, $\delta > 0$, and $|\arg z| < \frac{\pi}{2} - \delta$. Then*

$$e^{-z} = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} z^{-t} \Gamma(t) dt,$$

where the limits of integration define the contour shown in Figure 1.

Applying Proposition 4 to the first term in (10), we have

$$\sum_{\lambda \in \Lambda} \frac{\lambda}{e^{a\lambda} - 1} = \sum_{\lambda \in \Lambda} \sum_{l \in \mathbb{N}} \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} a^{-t} \lambda^{-(t-1)} l^{-t} \Gamma(t) dt,$$

which gives

$$\sum_{\lambda \in \Lambda} \frac{\lambda}{e^{a\lambda} - 1} = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} a^{-t} \zeta_{\Lambda}(t-1) \zeta_{\mathcal{R}}(t) \Gamma(t) dt,$$

where the contour is similar to Figure 1, but $\sigma > \mu_0 + 1$ so that all of the residues of the integrand lie to the left of σ . This range for σ guarantees absolute convergence of the resulting series and allows for an interchange of summation and integration. In the right half-plane, the integrand has simple poles at $t = \mu_0 + 1, \mu_1 + 1, \dots, 1, 0$, and we find

$$(11) \quad \sum_{\lambda \in \Lambda} \frac{\lambda}{e^{a\lambda} - 1} = \sum_{i=0}^d \frac{\zeta_{\mathcal{R}}(\mu_i + 1)}{a^{\mu_i + 1}} \mu_i A_{-\mu_i} + \frac{A_0}{a} + o\left(\frac{1}{a}\right),$$

where the A_k 's are defined by (3). The term $o(1/a)$ summarizes subleading contributions as $a \rightarrow 0$, the leading one of those behaving like $1/a^{1-\varepsilon}$, $\varepsilon > 0$, ε depending on the location of the right most pole of $\zeta_{\Lambda}(s)$ on the negative real axis.

The analysis of the second term in (10) follows along the same lines. For $k = 1$, using again Proposition 4, we write

$$(12) \quad S_\Lambda(e^{-a}) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} a^{-t} \zeta_\Lambda(t) \zeta_{\mathcal{R}}(t) \Gamma(t) dt.$$

For $k > 1$ we need to focus on the terms $\vartheta^{s-1} S_\Lambda(e^{-a})$ for $s > 1$. Note that (8) gives

$$(13) \quad \vartheta^{s-1} S_\Lambda(x) = \sum_{\lambda \in \Lambda} \frac{\sum_{j=1}^s c_j^{(s)} x^{j\lambda}}{(1-x^\lambda)^s} = \sum_{j=1}^s c_j^{(s)} \sum_{\lambda \in \Lambda} \frac{x^{j\lambda}}{(1-x^\lambda)^s} = \sum_{j=1}^s c_j^{(s)} S_\Lambda^{s,j}(x),$$

where $S_\Lambda^{s,j}(x) = \sum_{\lambda \in \Lambda} \frac{x^{j\lambda}}{(1-x^\lambda)^s}$, and the $c_j^{(s)}$ are defined¹ as in [29] by $c_0^{(s)} = 0$, $c_1^{(1)} = 1$, $c_1^{(2)} = 1$, $c_2^{(2)} = 0$, and for $s \geq 2$,

$$c_j^{(s+1)} = \begin{cases} jc_j^{(s)} + (s-j+1)c_{j-1}^{(s)}, & 1 \leq j \leq s \\ 0, & j = s+1. \end{cases}$$

Also, note the identity

$$(14) \quad \sum_{j=1}^s c_j^{(s)} = (s-1)!.$$

Using as before the substitution $x = e^{-a}$, we are interested in sums of the form

$$S_\Lambda^{s,j}(e^{-a}) = \sum_{\lambda \in \Lambda} \frac{e^{-aj\lambda}}{(1-e^{-a\lambda})^s}.$$

A little arithmetic yields

$$S_\Lambda^{s,j}(e^{-a}) = \sum_{\lambda \in \Lambda} \frac{e^{-aj\lambda}}{(1-e^{-a\lambda})^s} = \sum_{\lambda \in \Lambda} e^{-aj\lambda} \sum_{l \in \mathbb{N}_0} e_l^{(s)} e^{-a\lambda l} = \sum_{\lambda \in \Lambda} \sum_{l \in \mathbb{N}_0} e_l^{(s)} e^{-a\lambda(l+j)},$$

where $e_l^{(s)} = \binom{l+s-1}{s-1}$. Applying Proposition 4 gives

$$\begin{aligned} S_\Lambda^{s,j}(e^{-a}) &= \sum_{\lambda \in \Lambda} \sum_{l \in \mathbb{N}_0} e_l^{(s)} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} a^{-t} \lambda^{-t} (l+j)^{-t} \Gamma(t) dt \\ &= \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} a^{-t} \left(\sum_{\lambda \in \Lambda} \lambda^{-t} \right) \left(\sum_{l \in \mathbb{N}_0} e_l^{(s)} (l+j)^{-t} \right) \Gamma(t) dt. \end{aligned}$$

Since here we have to deal with Barnes zeta-functions of different dimension, we adopt the notation that $\zeta_{\mathcal{B}}^{(d)}$ is the Barnes zeta-function of dimension d . With this new notation we find, recalling Proposition 1,

$$(15) \quad S_\Lambda^{s,j}(e^{-a}) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} a^{-t} \zeta_\Lambda(t) \zeta_{\mathcal{B}}^{(s)}(t, j) \Gamma(t) dt.$$

Although the small- $|a|$ expansion of (12) and (15) can be, and will be, obtained later on, at this stage let us content ourselves with the following observation. The leading $a \rightarrow 0$ behavior of $S_\Lambda^{(k)}(e^{-a})$, for $l_k > 0$ suitable, is seen to be of the form

¹Correcting a typo in [29].

a^{-l_k} , respectively $a^{-l_k} \log a$, depending on the location of μ_0 . We therefore will have

$$(16) \quad \frac{\frac{d}{da} S_{\Lambda}^{(k)}(e^{-a})}{S_{\Lambda}^{(k)}(e^{-a})} = \frac{c_1(k) + c_2(k)/\log a}{a},$$

the numbers $c_1(k)$ and $c_2(k)$ depending on the location of μ_0 . In all cases, it is seen that the k -dependent correction to the saddle-point equation is of the order $\mathcal{O}(1/a)$. This will turn out to be of great importance for the analysis to follow.

Using the small- $|a|$ expansion displayed in (11) and (16), the saddle-point equation (10) allows us to uniquely determine a in terms of n , at least for large n .

With the saddle-point α_k known in terms of n , for large n , we determine asymptotic answers for the moments by applying a theorem of Olver (Thm 7.1, p. 126 of [28]) to (6). Noting that

$$\left. \frac{d^2}{da^2} \left(a + \frac{1}{n} \log \vartheta^k G_{\Lambda}(e^{-a}) \right) \right|_{a=\alpha_k} = - \left. \frac{1}{n} \frac{dn}{da} \right|_{a=\alpha_k},$$

this theorem takes the following form:

Lemma 1. *For α_k the solution of (10), as $n \rightarrow \infty$, we have*

$$(17) \quad t_{\Lambda}^k(n) = \frac{e^{n\alpha_k} \vartheta^k G_{\Lambda}(e^{-\alpha_k})}{2\pi} \cdot \left[\sqrt{\frac{-2\pi}{\left. \frac{dn}{da} \right|_{a=\alpha_k}}} + \mathcal{O}(n^{-3/2}) \right].$$

As indicated, α_k has to be thought of as being replaced by its large- n asymptotic expansion so that (17) represents a large- n asymptotic expansion. This will be explicitly done once we start looking at specific sequences Λ in Section 4.

The next important observation is that to leading order as $n \rightarrow \infty$, the result for the moments is *independent* of the k used for the saddle-point α_k .

Proposition 5. *For all k , as $n \rightarrow \infty$, we have*

$$t_{\Lambda}^k(n) = t_{\Lambda}^0(n) \cdot S_{\Lambda}^{(k)}(e^{-\alpha_0})[1 + o(1)].$$

Proof. Applying equation (17) amounts to evaluating $e^{na + \log G_{\Lambda}(e^{-a})}$, $S_{\Lambda}^{(k)}(e^{-a})$, and dn/da at the saddle-points $a = \alpha_k$. We first note that the saddle-point equation (10), together with (11) and (16), imply

$$\frac{1}{\alpha_k} = \left(\frac{n}{\zeta_{\mathcal{R}}(\mu_0 + 1) A_{-\mu_0} \mu_0} \right)^{\frac{1}{\mu_0 + 1}} [1 + o(1)],$$

where $o(1)$ denotes terms that vanish as $n \rightarrow \infty$. In particular, to leading order, $S_{\Lambda}^{(k)}(e^{-\alpha_k})$ and $dn/da|_{a=\alpha_k}$ are independent of the saddle-point used.

To show this independence for $e^{n\alpha_k + \log G_{\Lambda}(e^{-\alpha_k})}$ to the relevant order, considerably more work is necessary because of the exponential magnifying factor. In order to show the Proposition, we need to show that $n\alpha_k + \log G_{\Lambda}(e^{-\alpha_k}) = n\alpha_0 + \log G_{\Lambda}(e^{-\alpha_0}) + o(1)$ such that the difference due to the saddle-point chosen only produces subleading order corrections. First, again from equations (10), (11) and

(16),

$$\frac{1}{\alpha_k} = \left(\frac{n}{\zeta_{\mathcal{R}}(\mu_0 + 1)A_{-\mu_0}\mu_0} \right)^{\frac{1}{\mu_0+1}} \times \left(1 - \frac{1}{\mu_0 + 1} \frac{\tilde{c}_1(k) + c_2(k)/\log n}{\zeta_{\mathcal{R}}(\mu_0 + 1)A_{-\mu_0}\mu_0} \left(\frac{n}{\zeta_{\mathcal{R}}(\mu_0 + 1)A_{-\mu_0}\mu_0} \right)^{-\frac{\mu_0}{\mu_0+1}} + \dots \right),$$

where the leading k -dependence of the saddle-point solution has been depicted explicitly; $\tilde{c}_1(k)$ is determined from $c_1(k)$ and $c_2(k)$. Using the product representation of $G_{\Lambda}(e^{-a})$ from (5), we have

$$(18) \quad \log G_{\Lambda}(e^{-a}) = \sum_{\lambda \in \Lambda} \sum_{l \in \mathbb{N}} \frac{e^{-a\lambda l}}{l} = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} a^{-t} \zeta_{\Lambda}(t) \zeta_{\mathcal{R}}(t+1) \Gamma(t) dt.$$

Depicting only terms relevant for the leading k -dependence of the contributions considered, we have

$$\log G_{\Lambda}(e^{-\alpha_k}) = \alpha_k^{-\mu_0} \zeta_{\mathcal{R}}(\mu_0 + 1) A_{-\mu_0} + o(\alpha_k^{-\mu_0}).$$

Therefore, from equations (10) and (11), we conclude

$$\begin{aligned} n\alpha_k + \log G_{\Lambda}(e^{-\alpha_k}) &= \tilde{c}_1(k) + \frac{1}{\log n} c_2(k) + \alpha_k^{-\mu_0} (\mu_0 + 1) \zeta_{\mathcal{R}}(\mu_0 + 1) A_{-\mu_0} + \dots \\ &= 0 + \dots, \end{aligned}$$

that is, there is no k -dependence of this expression up to the order $o(1)$, which shows the assertion. \square

So in the following, saddle-point will always refer to $\alpha := \alpha_0$, which is a solution of

$$(19) \quad n = \sum_{\lambda \in \Lambda} \frac{\lambda}{e^{\alpha\lambda} - 1}.$$

To exploit equations (17) and (5) we first need a more complete expansion of equation (18). The relevant integrand has simple poles at $t = \mu_i$ for $i = 0, 1, \dots, d$, and a double pole at $t = 0$. Calculation of the integral gives, as $\alpha \rightarrow 0$,

$$\log G_{\Lambda}(e^{-\alpha}) = \sum_{i=0}^d \frac{\zeta_{\mathcal{R}}(\mu_i + 1)}{\alpha^{\mu_i}} A_{-\mu_i} - A_0 \log \alpha + \zeta'_{\Lambda}(0) + o(1).$$

Since $e^{n\alpha} G_{\Lambda}(e^{-\alpha}) = e^{n\alpha + \log G_{\Lambda}(e^{-\alpha})}$, we have the following lemma.

Lemma 2. *For the solution of (19),*

$$e^{n\alpha} G_{\Lambda}(e^{-\alpha}) = \alpha^{-A_0} \exp \left[\sum_{i=0}^d \frac{\mu_i + 1}{\alpha^{\mu_i}} \zeta_{\mathcal{R}}(\mu_i + 1) A_{-\mu_i} + A_0 + \zeta'_{\Lambda}(0) \right] [1 + o(1)].$$

Changing focus to $\frac{dn}{da}|_{a=\alpha}$, we note that this quantity is not exponentiated in the solution of $t_{\Lambda}^k(n)$. Thus, as mentioned, to obtain asymptotic results we need only determine the leading order. With this in mind, from equation (11), as $\alpha \rightarrow 0$,

$$-\frac{dn}{da} \Big|_{a=\alpha} = \frac{\mu_0(\mu_0 + 1)}{\alpha^{\mu_0+2}} \zeta_{\mathcal{R}}(\mu_0 + 1) A_{-\mu_0} [1 + o(1)],$$

so that

$$(20) \quad \frac{1}{2\pi} \left[\sqrt{\frac{-2\pi}{\frac{dn}{da}|_{a=\alpha}}} + \mathcal{O}(n^{-3/2}) \right] = (2\pi\mu_0(\mu_0 + 1)\zeta_{\mathcal{R}}(\mu_0 + 1)A_{-\mu_0})^{-\frac{1}{2}} \\ \times \alpha^{\frac{\mu_0}{2}+1} [1 + o(1)].$$

With the main calculations behind us, we now begin to consider specific k values, making two cases; $k = 0$ and $k \geq 1$. The proceeding calculation brings us to our first general moment theorem.

3.2. General Λ -type: $k = 0$. To evaluate the 0-th moment of the general Λ -type, $t_{\Lambda}^0(n)$, we apply (17), with $k = 0$, along with Lemma 2 and (20), to give the following theorem.

Theorem 1. *For Λ -type partitions,*

$$t_{\Lambda}^0(n) = (2\pi\mu_0(\mu_0 + 1)\zeta_{\mathcal{R}}(\mu_0 + 1)A_{-\mu_0})^{-\frac{1}{2}} \alpha^{\frac{\mu_0}{2}+1-A_0} \\ \times \exp \left[\sum_{i=0}^d \frac{\mu_i + 1}{\alpha^{\mu_i}} \zeta_{\mathcal{R}}(\mu_i + 1)A_{-\mu_i} + A_0 + \zeta'_{\Lambda}(0) \right] [1 + o(1)],$$

where α is the solution of (11) and the A_i 's are defined by (3).

Once α is replaced by its large- n asymptotic expansion, this determines the large- n asymptotic expansion of $t_{\Lambda}^0(n)$.

Let us recall that $t_{\Lambda}^0(n)$ gives the asymptotic result for the number of partitions of an integer n over the sequence Λ .

3.3. General Λ -type: Case $k \geq 1$. To evaluate the k -th moment of the general Λ -type for $k \geq 1$, $t_{\Lambda}^k(n)$, note that we need only evaluate $S_{\Lambda}^{(k)}(e^{-\alpha})$ and then apply Proposition 5.

It is evident from (9) that within the calculation for general $k \geq 1$, we will need the specific calculation for $\vartheta^0 S_{\Lambda}(e^{-\alpha}) = S_{\Lambda}(e^{-\alpha})$.

Using Proposition 4,

$$S_{\Lambda}(e^{-\alpha}) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \alpha^{-t} \zeta_{\Lambda}(t) \zeta_{\mathcal{R}}(t) \Gamma(t) dt.$$

To evaluate $S_{\Lambda}(e^{-\alpha})$ we must distinguish between a few cases. The above integral has different values depending on whether $\mu_0 < 1$, $\mu_0 = 1$, or $\mu_0 > 1$. We treat these cases independently.

For $\mu_0 < 1$, the leading pole is the simple pole of $\zeta_{\mathcal{R}}(t)$ at $t = 1$, and so

$$(21) \quad S_{\Lambda}(e^{-\alpha}) = \frac{\zeta_{\Lambda}(1)}{\alpha} [1 + \mathcal{O}(\alpha^{1-\mu_0})].$$

For $\mu_0 = 1$, the integrand has a double pole at $t = 1$, and

$$(22) \quad S_{\Lambda}(e^{-\alpha}) = \frac{1}{\alpha} \left(\text{FP}_{t=1} \{ \zeta_{\Lambda}(t) \} - A_{-1} \log \alpha \right) [1 + \mathcal{O}(\alpha^{1-\mu_1})],$$

where $\text{FP}_{t=1} \{ \zeta_{\Lambda}(t) \}$ denotes the finite part (or constant term) of the expansion of $\zeta_{\Lambda}(t)$ around $t = 1$.

For $\mu_0 > 1$, the integrand has a simple pole at $t = \mu_0$, and

$$(23) \quad S_\Lambda(e^{-\alpha}) = \frac{\zeta_{\mathcal{R}}(\mu_0)A_{-\mu_0}}{\alpha^{\mu_0}} \left[1 + \mathcal{O}\left(\alpha^{\min(\mu_0-1, \mu_0-\mu_1)}\right) \right].$$

Having completed the evaluation of $S_\Lambda(e^{-\alpha})$, we now focus on the terms $\vartheta^{s-1}S_\Lambda(e^{-\alpha})$ for $s > 1$. As shown previously, see (13) and (15), the relevant quantity to consider is

$$S_\Lambda^{s,j}(e^{-\alpha}) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \alpha^{-t} \zeta_\Lambda(t) \zeta_{\mathcal{B}}^{(s)}(t, j) \Gamma(t) dt.$$

Note that we need k values for s , namely $s = 1, 2, \dots, k$. Previously in this section we considered $s = 1$, now we turn our attention to the following three cases for $s \geq 2$: $s < \mu_0$, $s = \mu_0$, and $s > \mu_0$.

Let us remark, that although for each given case better error terms could be given, in the generality considered there are many cases necessary. Therefore, in order to make everything more readable, we refrain from doing so.

For $s < \mu_0$, the leading term comes from the simple pole of $\zeta_\Lambda(t)$ at $t = \mu_0$, so that

$$S_\Lambda^{s,j}(e^{-\alpha}) = \frac{\zeta_{\mathcal{B}}^{(s)}(\mu_0, j)A_{-\mu_0}}{\alpha^{\mu_0}} [1 + o(1)].$$

Now we have, using the above and (13),

$$\vartheta^{s-1}S_\Lambda(e^{-\alpha}) = \alpha^{-\mu_0} A_{-\mu_0} \sum_{j=1}^s c_j^{(s)} \zeta_{\mathcal{B}}^{(s)}(\mu_0, j) [1 + o(1)].$$

For $s = \mu_0$, the leading term comes from the double pole of $\zeta_\Lambda(t) \zeta_{\mathcal{B}}^{(s)}(t, j)$ at $t = \mu_0$. From (1) it is immediate that

$$\operatorname{Res}_{t=s} \left\{ \zeta_{\mathcal{B}}^{(s)}(t, j) \right\} = \frac{1}{\Gamma(s)},$$

and so

$$S_\Lambda^{s,j}(e^{-\alpha}) = \frac{1}{\alpha^s} \left(\operatorname{FP}_{t=s} \{ \zeta_\Lambda(t) \} + A_{-s} \operatorname{FP}_{t=s} \left\{ \zeta_{\mathcal{B}}^{(s)}(t, j) \right\} + \frac{A_{-s}}{\Gamma(s)} (\psi(s) - \log \alpha) \right) [1 + o(1)].$$

Thus, with (14),

$$\begin{aligned} \vartheta^{s-1}S_\Lambda(e^{-\alpha}) &= \frac{1}{\alpha^s} \left((s-1)! \cdot \operatorname{FP}_{t=s} \{ \zeta_\Lambda(t) \} + A_{-s} (\psi(s) - \log \alpha) + \right. \\ &\quad \left. + \sum_{j=1}^s c_j^{(s)} \operatorname{FP}_{t=s} \left\{ \zeta_{\mathcal{B}}^{(s)}(t, j) \right\} \right) [1 + o(1)]. \end{aligned}$$

Finally, for $s > \mu_0$, the leading term comes from the simple pole of $\zeta_{\mathcal{B}}^{(s)}(t, j)$ at $t = s$, so that

$$S_\Lambda^{s,j}(e^{-\alpha}) = \alpha^{-s} \zeta_\Lambda(s) [1 + o(1)],$$

and so

$$(24) \quad \vartheta^{s-1}S_\Lambda(e^{-\alpha}) = (s-1)! \cdot \alpha^{-s} \zeta_\Lambda(s) [1 + o(1)],$$

where again (14) has been used.

We are now in a position to evaluate $S_\Lambda^{(k)}(e^{-\alpha})$ as defined in (9); for the meaning of the notation Σ followed by a k -component expression consult the paragraph below (9). Again we must consider three cases: $\mu_0 < 1$, $\mu_0 = 1$, and $\mu_0 > 1$.

If $\mu_0 < 1$, (21) and (24) give

$$S_\Lambda^{(k)}(e^{-\alpha}) = \alpha^{-k} \sum (\zeta_\Lambda(1), \zeta_\Lambda(2), 2\zeta_\Lambda(3), \dots, (k-1)!\zeta_\Lambda(k)) [1 + o(1)].$$

If $\mu_0 = 1$, we have directly from (22) and (24) that

$$S_\Lambda^{(k)}(e^{-\alpha}) = \alpha^{-k} \sum_{t=1}^k \left(\text{FP} \{ \zeta_\Lambda(t) \} - A_{-1} \log \alpha, \zeta_\Lambda(2), 2\zeta_\Lambda(3), \dots, (k-1)!\zeta_\Lambda(k) \right) \times [1 + o(1)].$$

Finally, if $\mu_0 > 1$, then the leading term of $S_\Lambda^{(k)}(e^{-\alpha})$ comes from $b_1 = k$. This implies

$$S_\Lambda^{(k)}(e^{-\alpha}) = (S_\Lambda(e^{-\alpha}))^k [1 + o(1)],$$

so that (23) gives

$$S_\Lambda^{(k)}(e^{-\alpha}) = \alpha^{-\mu_0 k} A_{-\mu_0}^k [\zeta_{\mathcal{R}}(\mu_0)]^k [1 + o(1)].$$

We may now apply Proposition 5 to give the following theorem.

Theorem 2. *For $k \geq 1$, and α the solution of (11), the following results hold.*

(i) *If $\mu_0 > 1$, then*

$$t_\Lambda^k(n) = t_\Lambda^0(n) \cdot \alpha^{-\mu_0 k} A_{-\mu_0}^k [\zeta_{\mathcal{R}}(\mu_0)]^k [1 + o(1)].$$

(ii) *If $\mu_0 = 1$, then*

$$t_\Lambda^k(n) = t_\Lambda^0(n) \cdot \alpha^{-k} \sum_{t=1}^k \left(\text{FP} \{ \zeta_\Lambda(t) \} - A_{-1} \log \alpha, \zeta_\Lambda(2), 2\zeta_\Lambda(3), \dots, (k-1)!\zeta_\Lambda(k) \right) [1 + o(1)].$$

(iii) *If $\mu_0 < 1$, then*

$$t_\Lambda^k(n) = t_\Lambda^0(n) \cdot \alpha^{-k} \sum (\zeta_\Lambda(1), \zeta_\Lambda(2), 2\zeta_\Lambda(3), \dots, (k-1)!\zeta_\Lambda(k)) [1 + o(1)].$$

We call Theorems 1 and 2, the General Moment Theorems.

For $k = 1$, the above theorem yields the following corollary.

Corollary 2. *For Λ -type partitions, with α the solution of (11) and $t_\Lambda^0(n)$ as given in Theorem 1, we have:*

(i) *For $\mu_0 < 1$,*

$$t_\Lambda^1(n) = t_\Lambda^0(n) \cdot \frac{\zeta_\Lambda(1)}{\alpha} [1 + o(1)].$$

(ii) *For $\mu_0 = 1$,*

$$t_\Lambda^1(n) = t_\Lambda^0(n) \cdot \frac{1}{\alpha} \left(\text{FP} \{ \zeta_\Lambda(t) \} - A_{-1} \log \alpha \right) [1 + o(1)].$$

(iii) *For $\mu_0 > 1$,*

$$t_\Lambda^1(n) = t_\Lambda^0(n) \cdot \frac{\zeta_{\mathcal{R}}(\mu_0) A_{-\mu_0}}{\alpha^{\mu_0}} [1 + o(1)].$$

3.4. Expected number of summands. In this section we give expressions for the expected number of summands.

Definition 7. The expected number of summands of a Λ -type partition of an integer n , denoted by $m_\Lambda(n)$, is defined as

$$m_\Lambda(n) = \frac{t_\Lambda^1(n)}{t_\Lambda^0(n)}.$$

Lemma 3. *The expected number of summands of a Λ -type partition of an integer n is*

$$m_\Lambda(n) = S_\Lambda(e^{-\alpha}) [1 + o(1)],$$

where α is the solution of (11).

Proof. This result is an immediate consequence of Proposition 5. \square

In light of Lemma 3, a direct corollary of equations (21), (22), and (23), is the following result.

Theorem 3. *For α the solution of (11), the following assertions hold.*

(i) *If $\mu_0 < 1$, then*

$$m_\Lambda(n) = \frac{\zeta_\Lambda(1)}{\alpha} [1 + o(1)].$$

(ii) *If $\mu_0 = 1$, then*

$$m_\Lambda(n) = \frac{1}{\alpha} \left(\text{FP}_{t=1} \{ \zeta_\Lambda(t) \} - A_{-1} \log \alpha \right) [1 + o(1)].$$

(iii) *If $\mu_0 > 1$, then*

$$m_\Lambda(n) = \frac{\zeta_{\mathcal{R}}(\mu_0) A_{-\mu_0}}{\alpha^{\mu_0}} [1 + o(1)].$$

4. APPLICATIONS

In this Section we apply the General Moment Theorems of the previous section to a variety of special cases. We first reproduce the results of Hardy and Ramanujan. Then we will examine the case of one singularity at $\mu > 0$. Furthermore, we examine multidimensional applications in Barnes and Epstein type sequences, where in each case we calculate higher moments and the expected number of summands. The results frequently use the notation Σ followed by a k -component expression. We remind the reader that this notation is explained below (9).

4.1. Hardy and Ramanujan. Let $\Lambda = \mathbb{N}$ such that $\zeta_\Lambda(s) = \zeta_{\mathcal{R}}(s)$. This is the case with one singularity at $\mu = 1$ and we find the very familiar results of Hardy and Ramanujan [17] for the asymptotic number of ways to express an integer n as the sum of lesser integers. The following theorems follow immediately from the General Moment Theorems, see also [29].

Lemma 4. *For sufficiently large n ,*

$$\alpha = \frac{\pi}{(6n)^{\frac{1}{2}}} - \frac{1}{4n} + \mathcal{O}(n^{-\frac{3}{2}}).$$

Theorem 4 (Hardy and Ramanujan [17]). *The asymptotic number of ways to partition n over \mathbb{N} is*

$$t_0(n) = \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{\frac{2n}{3}}} \left[1 + \mathcal{O}(n^{-\frac{1}{2}}) \right].$$

Corollary 3. *Let $k = 1$, then*

$$t_1(n) = \frac{\sqrt{2}}{4\pi\sqrt{n}} e^{\pi\sqrt{\frac{2n}{3}}} \left(\gamma + \log \frac{\sqrt{6n}}{\pi} \right) \left[1 + \mathcal{O}(n^{-1/2}) \right].$$

Corollary 4. *For $k \geq 2$,*

$$t_k(n) = \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{\frac{2n}{3}}} \left(\frac{\sqrt{6n}}{\pi} \right)^k \times \\ \times \sum \left(\gamma + \log \frac{\sqrt{6n}}{\pi}, \zeta_{\mathcal{R}}(2), 2\zeta_{\mathcal{R}}(3), \dots, (k-1)!\zeta_{\mathcal{R}}(k) \right) \left[1 + \mathcal{O}(n^{-\frac{1}{2}}) \right].$$

Corollary 5. *The expected number of summands of a Riemann type partition of an integer n is*

$$m(n) = \frac{\sqrt{6n}}{\pi} \left(\gamma + \log \frac{\sqrt{6n}}{\pi} \right) \left[1 + \mathcal{O}(n^{-\frac{1}{2}}) \right].$$

4.2. Zeta-functions with one singularity. Let us suppose that we have a non-decreasing sequence of natural numbers Λ , whose corresponding partition function $\Theta(t)$ admits the full asymptotic expansion

$$\Theta(t) = \sum_{i=0}^{\infty} A_{k_i} t^{k_i},$$

in which $k_0 < 0$, and $k_i \geq 0$ for all $i > 0$. Define $\mu = -k_0$. Then the zeta-function associated with the sequence Λ has only one singularity on the positive real axis, namely at $t = \mu$. Having met the assumptions of the General Moment Theorems, we proceed to apply them to the above sequence Λ .

Note that within the previous section we could not easily solve for n in terms of the saddle-point α . Since we now have only one singularity, the leading orders can be determined easily and (11) gives

$$\alpha = \left(\frac{\zeta_{\mathcal{R}}(\mu+1)\mu A_{-\mu}}{n} \right)^{\frac{1}{\mu+1}} + \frac{A_0}{\mu+1} \frac{1}{n} + o(1/n)$$

The General Moment Theorems now yield the following corollaries.

Corollary 6. *For sequences Λ as described, we have*

$$t_{\Lambda}^0(n) = (2\pi(\mu+1))^{-\frac{1}{2}} (\mu\zeta_{\mathcal{R}}(\mu+1)A_{-\mu})^{\frac{1-2A_0}{2(\mu+1)}} n^{\frac{2A_0-2-\mu}{2(\mu+1)}} \times \\ \times \exp \left[\left(\frac{n}{\mu} \right)^{\frac{\mu}{\mu+1}} (\mu+1) (\zeta_{\mathcal{R}}(\mu+1)A_{-\mu})^{\frac{1}{\mu+1}} + \zeta'_{\Lambda}(0) \right] [1 + o(1)].$$

For the higher moments, according to Theorem 2, we need to distinguish three cases.

Corollary 7. *For sequences Λ as described, we have for $k \geq 1$ the following.*

(i) For $\mu > 1$,

$$t_{\Lambda}^k(n) = t_{\Lambda}^0(n) \cdot \left(\frac{n}{\mu \zeta_{\mathcal{R}}(\mu+1)} \right)^{\frac{\mu k}{\mu+1}} A_{-\mu}^{\frac{k}{\mu+1}} [\zeta_{\mathcal{R}}(\mu)]^k [1 + o(1)].$$

(ii) For $\mu = 1$,

$$\begin{aligned} t_{\Lambda}^k(n) &= t_{\Lambda}^0(n) \cdot n^{\frac{k}{2}} \left(\frac{\pi^2 A_{-1}}{6} \right)^{\frac{-k}{2}} \\ &\times \sum \left(\text{FP}_{t=1} \{ \zeta_{\Lambda}(t) \} - \frac{A_{-1}}{2} \log \left(\frac{\pi^2 A_{-1}}{6n} \right), \zeta_{\Lambda}(2), 2\zeta_{\Lambda}(3), \dots, (k-1)! \zeta_{\Lambda}(k) \right) \\ &\times [1 + o(1)] \end{aligned}$$

(iii) For $\mu < 1$,

$$\begin{aligned} t_{\Lambda}^k(n) &= t_{\Lambda}^0(n) \cdot n^{\frac{k}{\mu+1}} (\zeta_{\mathcal{R}}(\mu+1) \mu A_{-\mu})^{\frac{-k}{\mu+1}} \\ &\times \sum (\zeta_{\Lambda}(1), \zeta_{\Lambda}(2), 2\zeta_{\Lambda}(3), \dots, (k-1)! \zeta_{\Lambda}(k)) [1 + o(1)]. \end{aligned}$$

Applying Theorem 3 gives the following corollary.

Corollary 8. *For sequences Λ as described, the following hold.*

(i) If $\mu < 1$, then

$$m_{\Lambda}(n) = \zeta_{\Lambda}(1) \left(\frac{n}{\zeta_{\mathcal{R}}(\mu+1) \mu A_{-\mu}} \right)^{\frac{1}{\mu+1}} [1 + o(1)].$$

(ii) If $\mu = 1$, then

$$m_{\Lambda}(n) = \left(\frac{6n}{\pi^2 A_{-1}} \right)^{\frac{1}{2}} \left(\text{FP}_{t=1} \{ \zeta_{\Lambda}(t) \} - \frac{A_{-1}}{2} \log \frac{\pi^2 A_{-1}}{6n} \right) [1 + o(1)].$$

(iii) If $\mu > 1$, then

$$m_{\Lambda}(n) = \zeta_{\mathcal{R}}(\mu) A_{-\mu}^{\frac{1}{\mu+1}} \left(\frac{n}{\zeta_{\mathcal{R}}(\mu+1) \mu} \right)^{\frac{\mu}{\mu+1}} [1 + o(1)].$$

4.3. Barnes type moments. We will now consider two special cases of Barnes type moments, the two-dimensional and three-dimensional cases, with $\vec{r} = \vec{1}$ and $c = 0$. These results correspond to two and three-dimensional oscillator assemblies considered by Nanda [25].

4.3.1. Two-dimensional. For the two-dimensional case, (11) gives

$$\alpha = \left(\frac{2\zeta_{\mathcal{R}}(3)}{n} \right)^{\frac{1}{3}} + \left(\frac{\zeta_{\mathcal{R}}(2)}{3(2\zeta_{\mathcal{R}}(3))^{\frac{1}{3}}} \right) \frac{1}{n^{\frac{2}{3}}} - \frac{7}{36n} + \mathcal{O}\left(n^{-\frac{4}{3}}\right).$$

We use (1) to evaluate relevant residues and

$$\zeta'_{\mathcal{B}_2}(0, 0) = -\frac{1}{2} \log 2\pi + \zeta'_{\mathcal{R}}(-1).$$

We may now apply the General Moment Theorems to produce the following corollaries.

Corollary 9. *For the two-dimensional Barnes type sequence with $c = 0$ and $\vec{r} = \vec{1}$,*

$$t_{\mathcal{B}_2}^0(n) = \frac{(6\zeta_{\mathcal{R}}(3))^{-\frac{1}{2}}}{2\pi} \left(\frac{2\zeta_{\mathcal{R}}(3)}{n} \right)^{\frac{31}{36}} \\ \times \exp \left[\frac{3(\zeta_{\mathcal{R}}(3))^{\frac{1}{3}}}{2^{\frac{2}{3}}} n^{\frac{2}{3}} + \frac{\zeta_{\mathcal{R}}(2)}{2^{\frac{1}{3}}(\zeta_{\mathcal{R}}(3))^{\frac{1}{3}}} n^{\frac{1}{3}} - \frac{(\zeta_{\mathcal{R}}(2))^2}{12\zeta_{\mathcal{R}}(3)} + \zeta'_{\mathcal{R}}(-1) \right] \\ \times \left[1 + \mathcal{O}\left(n^{-\frac{1}{3}}\right) \right].$$

Making the substitution $n' = n/(2\zeta_{\mathcal{R}}(3))$, for ease of comparison, the above corollary corresponds to that of Nanda [25, Eq. (34)].

Corollary 10. *For the two-dimensional Barnes type sequence with $c = 0$, $\vec{r} = \vec{1}$, and $k \geq 1$,*

$$t_{\mathcal{B}_2}^k(n) = \frac{(6\zeta_{\mathcal{R}}(3))^{-\frac{1}{2}}}{2\pi} (\zeta_{\mathcal{R}}(2))^k \left(\frac{2\zeta_{\mathcal{R}}(3)}{n} \right)^{\frac{31-24k}{36}} \\ \times \exp \left[\frac{3(\zeta_{\mathcal{R}}(3))^{\frac{1}{3}}}{2^{\frac{2}{3}}} n^{\frac{2}{3}} + \frac{\zeta_{\mathcal{R}}(2)}{2^{\frac{1}{3}}(\zeta_{\mathcal{R}}(3))^{\frac{1}{3}}} n^{\frac{1}{3}} - \frac{(\zeta_{\mathcal{R}}(2))^2}{12\zeta_{\mathcal{R}}(3)} + \zeta'_{\mathcal{R}}(-1) \right] \\ \times \left[1 + \mathcal{O}\left(n^{-\frac{1}{3}}\right) \right].$$

Note that the case $k = 0$ of Corollary 10 is precisely $t_{\mathcal{B}_2}^0(n)$ of Corollary 9. In general this is true when the largest singularity is greater than 1; that is, in this paper, $\mu_0 > 1$. Thus when considering the three-dimensional Barnes case, we will give only the general result for $k \in \mathbb{N}_0$. But first we give the expected number of summands as a consequence of Theorem 3.

Corollary 11. *For the two-dimensional Barnes type partitions with $c = 0$ and $\vec{r} = \vec{1}$,*

$$m_{\mathcal{B}_2}(n) = \zeta_{\mathcal{R}}(2) \left(\frac{n}{2\zeta_{\mathcal{R}}(3)} \right)^{\frac{2}{3}} \left[1 + \mathcal{O}\left(n^{-\frac{1}{3}}\right) \right].$$

4.3.2. *Three-dimensional.* For the three-dimensional case, we evaluate (11) to give

$$\alpha = (3\zeta_{\mathcal{R}}(4))^{\frac{1}{4}} n^{-\frac{1}{4}} + \left(\frac{3^{\frac{1}{2}}\zeta_{\mathcal{R}}(3)}{4(\zeta_{\mathcal{R}}(4))^{\frac{1}{2}}} \right) n^{-\frac{1}{2}} \\ + \left(\frac{8\zeta_{\mathcal{R}}(2)\zeta_{\mathcal{R}}(4) - 3(\zeta_{\mathcal{R}}(3))^2}{3^{\frac{3}{4}}32(\zeta_{\mathcal{R}}(4))^{\frac{3}{4}}} \right) n^{-\frac{3}{4}} - \frac{5}{32n} + \mathcal{O}\left(n^{-\frac{5}{4}}\right).$$

Using (2.1) we evaluate relevant residues and

$$\zeta'_{\mathcal{B}_3}(0, 0) = -\frac{1}{2} \log 2\pi + \frac{3}{2} \zeta'_{\mathcal{R}}(-1) + \frac{1}{2} \zeta'_{\mathcal{R}}(-2).$$

We apply the General Moment Theorems to derive the following corollaries.

Corollary 12. *For the three-dimensional Barnes type sequence with $c = 0$, $\vec{r} = \vec{1}$, and $k \in \mathbb{N}_0$,*

$$t_{B_3}^k(n) = \frac{(3\zeta_{\mathcal{R}}(4))^{-\frac{1}{2}}}{4\pi} (\zeta_{\mathcal{R}}(3))^k \left(\frac{3\zeta_{\mathcal{R}}(4)}{n} \right)^{\frac{25-24k}{32}} \\ \times \exp \left[4\zeta_{\mathcal{R}}(4) \left(\frac{n}{3\zeta_{\mathcal{R}}(4)} \right)^{\frac{3}{4}} + \frac{3\zeta_{\mathcal{R}}(3)}{2} \left(\frac{n}{3\zeta_{\mathcal{R}}(4)} \right)^{\frac{1}{2}} \right. \\ \left. + \left(\zeta_{\mathcal{R}}(2) - \frac{3(\zeta_{\mathcal{R}}(3))^2}{8\zeta_{\mathcal{R}}(4)} \right) \left(\frac{n}{3\zeta_{\mathcal{R}}(4)} \right)^{\frac{1}{4}} + C \right] \left[1 + \mathcal{O}\left(n^{-\frac{1}{4}}\right) \right],$$

where $C = \frac{(\zeta_{\mathcal{R}}(3))^3}{8(\zeta_{\mathcal{R}}(4))^2} - \frac{\zeta_{\mathcal{R}}(2)\zeta_{\mathcal{R}}(3)}{4\zeta_{\mathcal{R}}(4)} + \frac{3}{2}\zeta'_{\mathcal{R}}(-1) + \frac{1}{2}\zeta'_{\mathcal{R}}(-2)$.

Setting $k = 0$, and making the substitution $n'' = n/(3\zeta_{\mathcal{R}}(4))$, the above corollary corresponds to that of Nanda [25, Eq. (51)].

Now, Theorem 3 yields the following corollary.

Corollary 13. *For the three-dimensional Barnes type partitions with $c = 0$ and $\vec{r} = \vec{1}$,*

$$m_{B_3}(n) = \zeta_{\mathcal{R}}(3) \left(\frac{n}{3\zeta_{\mathcal{R}}(4)} \right)^{\frac{3}{4}} \left[1 + \mathcal{O}\left(n^{-\frac{1}{4}}\right) \right].$$

4.4. Epstein type moments. We now consider the sequence $\Lambda = \{n_1^2 + n_2^2 + \dots + n_d^2 | \vec{n} \in \mathbb{N}_0^d - \{\vec{0}\}\}$. Using the resummation formula [18]

$$\sum_{l=-\infty}^{\infty} e^{-tl^2} = \sqrt{\frac{\pi}{t}} \sum_{l=-\infty}^{\infty} e^{-\frac{\pi^2}{t}l^2},$$

it is easily seen that the associated partition function has the small- t asymptotic expansion

$$\Theta_{\mathcal{E}}(t) \sim \sum_{n=0}^d A_{-\frac{n}{2}} t^{-\frac{n}{2}}$$

where the $A_{-\frac{n}{2}}$ are given by

$$A_0 = \frac{1}{2^d} - 1, \quad A_{-\frac{n}{2}} = \frac{1}{2^d} \binom{d}{n} \pi^{\frac{n}{2}}, \quad n = 1, \dots, d.$$

With the help of (4) this determines the relevant residues. Additional quantities needed in the corollaries stated below can be extracted from [12, 13, 14, 15, 22].

We now approach two and three-dimensional Epstein type moments in the same way that we considered the Barnes type.

4.4.1. Two-dimensional. Note that for the two-dimensional case, (11) gives

$$\alpha = \left(\frac{\pi\zeta_{\mathcal{R}}(2)}{4} \right)^{\frac{1}{2}} n^{-\frac{1}{2}} + \left(\frac{\pi^{\frac{1}{4}}\zeta_{\mathcal{R}}(\frac{3}{2})}{4\sqrt{2}(\zeta_{\mathcal{R}}(2))^{\frac{1}{4}}} \right) n^{-\frac{3}{4}} - \frac{3}{8n} + \mathcal{O}\left(n^{-\frac{5}{4}}\right).$$

Using this information we obtain the following corollaries to the General Moment Theorems.

Corollary 14. *For the two-dimensional Epstein type sequence with $c = 0$ and $\vec{r} = \vec{1}$,*

$$\begin{aligned} t_{\mathcal{E}_2}^0(n) &= (\pi^2 \zeta_{\mathcal{R}}(2))^{-\frac{1}{2}} \left(\frac{\pi \zeta_{\mathcal{R}}(2)}{4n} \right)^{\frac{9}{8}} \\ &\times \exp \left[\sqrt{\pi \zeta_{\mathcal{R}}(2)n} + \left(\frac{\pi^{\frac{1}{4}} \zeta_{\mathcal{R}}\left(\frac{3}{2}\right)}{4\sqrt{2}(\zeta_{\mathcal{R}}(2))^{\frac{1}{4}}} \right) n^{\frac{1}{4}} + \frac{(\zeta_{\mathcal{R}}\left(\frac{3}{2}\right))^2}{32\zeta_{\mathcal{R}}(2)} + \zeta'_{\mathcal{E}_2}(0) \right] \\ &\times \left[1 + \mathcal{O}\left(n^{-\frac{1}{4}}\right) \right]. \end{aligned}$$

Corollary 15. *For the two-dimensional Epstein type sequence with $c = 0$, $\vec{r} = \vec{1}$, and $k \geq 1$,*

$$\begin{aligned} t_{\mathcal{E}_2}^k(n) &= (\pi^2 \zeta_{\mathcal{R}}(2))^{-\frac{1}{2}} \left(\frac{\pi \zeta_{\mathcal{R}}(2)}{4n} \right)^{\frac{9-4k}{8}} \\ &\times \exp \left[\sqrt{\pi \zeta_{\mathcal{R}}(2)n} + \left(\frac{\pi^{\frac{1}{4}} \zeta_{\mathcal{R}}\left(\frac{3}{2}\right)}{4\sqrt{2}(\zeta_{\mathcal{R}}(2))^{\frac{1}{4}}} \right) n^{\frac{1}{4}} + \frac{(\zeta_{\mathcal{R}}\left(\frac{3}{2}\right))^2}{32\zeta_{\mathcal{R}}(2)} + \zeta'_{\mathcal{E}_2}(0) \right] \\ &\times \sum \left(\text{FP}_{t=1} \{ \zeta_{\mathcal{E}_2}(t) \} - \frac{\pi}{8} \log \left(\frac{\pi \zeta_{\mathcal{R}}(2)}{4n} \right), \zeta_{\mathcal{E}_2}(2), 2\zeta_{\mathcal{E}_2}(3), \dots, (k-1)!\zeta_{\mathcal{E}_2}(k) \right) \\ &\times \left[1 + \mathcal{O}\left(n^{-\frac{1}{4}}\right) \right]. \end{aligned}$$

Theorem 3 gives the expected number of summands.

Corollary 16. *For the two-dimensional Epstein type sequence with $c = 0$ and $\vec{r} = \vec{1}$,*

$$m_{\mathcal{E}_2}(n) = \left(\frac{4n}{\pi \zeta_{\mathcal{R}}(2)} \right)^{\frac{1}{2}} \left(\text{FP}_{t=1} \{ \zeta_{\mathcal{E}_2}(t) \} - \frac{\pi}{8} \log \left(\frac{\pi \zeta_{\mathcal{R}}(2)}{4n} \right) \right) \left[1 + \mathcal{O}\left(n^{-\frac{1}{4}}\right) \right].$$

4.4.2. *Three-dimensional.* For the three-dimensional Epstein case, (11) gives

$$\begin{aligned} \alpha &= \left(\frac{3\pi^{\frac{3}{2}} \zeta_{\mathcal{R}}\left(\frac{5}{2}\right)}{16} \right)^{\frac{2}{5}} n^{-\frac{2}{5}} + \left(\frac{3^{\frac{3}{5}} \zeta_{\mathcal{R}}(2)}{5} \right) \left(\frac{\pi}{2\zeta_{\mathcal{R}}\left(\frac{5}{2}\right)} \right)^{\frac{2}{5}} n^{-\frac{3}{5}} \\ &+ \frac{3^{\frac{4}{5}} \pi^{\frac{1}{5}}}{2^{\frac{1}{5}} 100 (\zeta_{\mathcal{R}}\left(\frac{5}{2}\right))^{\frac{6}{5}}} \left(5\zeta_{\mathcal{R}}\left(\frac{3}{2}\right) \zeta_{\mathcal{R}}\left(\frac{5}{2}\right) - 2(\zeta_{\mathcal{R}}(2))^2 \right) n^{-\frac{4}{5}} - \frac{7}{20n} \mathcal{O}\left(n^{-\frac{6}{5}}\right). \end{aligned}$$

Here we may note that $\mu_0 = \frac{3}{2} > 1$, so that we have the following corollary to the General Moment Theorems.

Corollary 17. *For the three-dimensional Epstein type sequence with $c = 0$, $\vec{r} = \vec{1}$, and $k \in \mathbb{N}_0$,*

$$\begin{aligned} t_{\mathcal{E}_3}^k(n) = & \left(\frac{15}{16} \pi^{\frac{5}{2}} \zeta_{\mathcal{R}} \left(\frac{5}{2} \right) \right)^{-\frac{1}{2}} \left(\frac{3\pi^{\frac{3}{2}} \zeta_{\mathcal{R}} \left(\frac{5}{2} \right)}{16n} \right)^{\frac{27-12k}{20}} \left(\frac{\pi^{\frac{3}{2}}}{8} \zeta_{\mathcal{R}} \left(\frac{3}{2} \right) \right)^k \\ & \times \exp \left[\left(\frac{5\pi^{\frac{3}{5}} (\zeta_{\mathcal{R}} \left(\frac{5}{2} \right))^{\frac{2}{5}}}{6^{\frac{3}{5}} 2} \right) n^{\frac{3}{5}} + \frac{3^{\frac{3}{5}} \zeta_{\mathcal{R}}(2)}{2} \left(\frac{\pi}{2\zeta_{\mathcal{R}} \left(\frac{5}{2} \right)} \right)^{\frac{2}{5}} n^{\frac{2}{5}} \right. \\ & + \frac{3^{\frac{4}{5}} \pi^{\frac{1}{5}}}{2^{\frac{1}{5}} 20 (\zeta_{\mathcal{R}} \left(\frac{5}{2} \right))^{\frac{6}{5}}} \left(5\zeta_{\mathcal{R}} \left(\frac{3}{2} \right) \zeta_{\mathcal{R}} \left(\frac{5}{2} \right) - 2(\zeta_{\mathcal{R}}(2))^2 \right) n^{\frac{1}{5}} \\ & \left. + \frac{(\zeta_{\mathcal{R}}(2))^3}{10 (\zeta_{\mathcal{R}} \left(\frac{5}{2} \right))^2} - \frac{3\zeta_{\mathcal{R}} \left(\frac{3}{2} \right) \zeta_{\mathcal{R}}(2)}{20\zeta_{\mathcal{R}} \left(\frac{5}{2} \right)} + \zeta'_{\mathcal{E}_3}(0) \right] \left[1 + \mathcal{O} \left(n^{-\frac{1}{5}} \right) \right]. \end{aligned}$$

Theorem 3 yields the following corollary.

Corollary 18. *For the three-dimensional Epstein type sequence with $c = 0$ and $\vec{r} = \vec{1}$,*

$$m_{\mathcal{E}_3}(n) = \frac{\pi^{\frac{3}{2}} \zeta_{\mathcal{R}} \left(\frac{3}{2} \right)}{8} \left(\frac{16n}{3\pi^{\frac{3}{2}} \zeta_{\mathcal{R}} \left(\frac{5}{2} \right)} \right)^{\frac{3}{5}} \left[1 + \mathcal{O} \left(n^{-\frac{1}{5}} \right) \right].$$

5. CONCLUSION

In this article we have developed a systematic approach to obtain, for large n , the moments of the number of partitions of n into m summands from a given sequence Λ of natural numbers. Under suitable assumptions on Λ , see (3), the moments are expressed in terms of information obtained from the small- t asymptotic expansion of the associated partition function. The main technical tool is the saddle-point method in combination with the application of (4).

Whenever the small- t asymptotic expansion of the partition function can be easily determined, the large- n expansions of the moments can be easily found. This has been demonstrated in Section 4 with sequences given by linear and quadratic sums of integers.

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